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1988 J. Phys. A: Math. Gen. 21 L311

(http://iopscience.iop.org/0305-4470/21/5/008)

J. Phys. A: Math. Gen. 20 (1988) L311-L314. Printed in the UK

LETTER TO THE EDITOR

Complete devil's staircase in an Ising model with competing interactions

Tânia Tomé and S R Salinas

Instituto de Física, Universidade de São Paulo, Caixa Postal 20516, 01498 São Paulo, SP, Brazil

Received 3 December 1987

Abstract. An Ising model on a regular Cayley tree, with competing ferro- and antiferromagnetic nearest-neighbour interactions, can be formulated as a discrete two-dimensional mapping. We use this mapping to obtain sequences of modulated phases, at low temperatures, associated with complete devil's staircases. The fractal dimensionality of the staircases increases with temperature. At higher temperatures the incommensurate phases may occupy regions of finite measure of the phase diagram.

An Ising model on a regular Cayley tree with nearest-neighbour competing interactions has been proposed to mimic the behaviour of spin glasses (Morita 1983, Horiguchi and Morita 1983, 1984). In a recent publication, de Oliveira and Salinas (1985) formulated this problem as a two-dimensional discrete mapping and performed some calculations, in the mean-field or infinite coordination limit, to obtain the corresponding phase diagrams. The lack of disorder precludes the appearance of a true spin-glass phase. However, the frustration introduced by the competing interactions turns out to be responsible for the existence of a sequence of modulated structures, at low temperatures, in a region of the phase diagram otherwise occupied by a spin-glass phase. In this letter, we report a detailed study of the devil's staircases corresponding to this sequence of modulated phases. In particular, we calculate the Hausdorff dimensionality, D, associated with the intervals which are not occupied by commensurate structures. At low temperatures, as $D<1$, the devil's staircases are complete. At higher temperatures, however, as $D=1$, the incommensurate phases occupy regions of finite measure in the phase diagram.

Let us consider a nearest-neighbour spin-$\frac{1}{2}$ Ising model on a regular Cayley tree of coordination z. In a simple version of the Morita model, each site is connected to n neighbours by bonds of type 1, with an exchange interaction $J_1=+J$, and to $z-n$ neighbours by bonds of type 2, with exchange $J_2=-J$. In the infinite coordination limit, we make $z\to\infty$, $n\to\infty$ and $J\to 0$, such that the parameters $p=(2n-z)/z^{1/2}$ and $\tilde{J}=Jz^{1/2}$ remain finite. The phase diagrams are drawn in terms of the temperature T, in units of $\tilde{J}/k_B$, where k_B is Boltzmann's constant, and the parameter p, which gives an indication about the ferro- or antiferromagnetic character of the overall interactions. According to de Oliveira and Salinas (1985), in this limit the recursion relations are given by

$$m_{l+1}=\tanh\left(\frac{p}{T}m_l-\frac{1}{T^2}q_l\right) \tag{1}$$

and

$$q_{l+1} = m_l \operatorname{sech}^2\left(\frac{p}{T} m_l - \frac{1}{T^2} q_l\right) \tag{2}$$

where the subscript l labels a shell of the tree. The variables m_l and q_l are given in terms of $m_l^{(1)}$ and $m_l^{(2)}$, which are associated with the effective fields induced on the spins of the innermost shells by bonds of types 1 and 2 respectively.

The phase diagram obtained from (1) and (2) is shown in figure 1. The paramagnetic and the ferromagnetic regions are associated with the trivial one-cycle fixed points, $m^* = 0$ and $m^* = \pm m_0 \neq 0$, respectively. The antiferromagnetic region is associated with a two-cycle fixed point. The commensurate phases are associated with periodic fixed points of the mapping and are thus characterised by a rational wavenumber q, where $q/2\pi$ is in the interval $[0, \frac{1}{2}]$. A plot of the wavenumber q against the parameter p, at a given temperature, shows a collection of steps of different widths and behaves as a devil's staircase (see figure 2). The steps of this staircase correspond to the regions occupied by the commensurate phases. As there is an infinite number of these structures, it is always possible to increase the accuracy of the calculations to find additional plateaux in the interstices between any two wider steps. It remains to be shown whether there is still room of finite measure for the presence of incommensurate phases.

At the paramagnetic border the critical wavenumber varies smoothly with the parameter p. It is then possible to show that, at least immediately below this border, the incommensurate phases occupy finite regions of the phase diagram. To search for

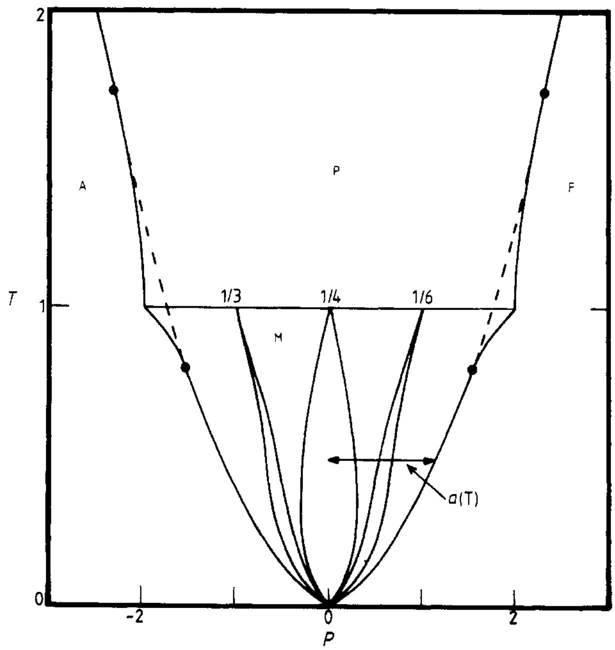

Figure 1. Global phase diagram in the limit of infinite coordination number. Paramagnetic (P), ferromagnetic (F), antiferromagnetic (A) and modulated (M) regions are shown. In the M region only a few main commensurate phases are shown. The F and A regions extend up to the broken line overlapping the M and P regions. The four dots are tricritical points. The interval $a(T)$, from $p = 0$ to the border of the ferromagnetic phase, is shown for $T = 0.5$.

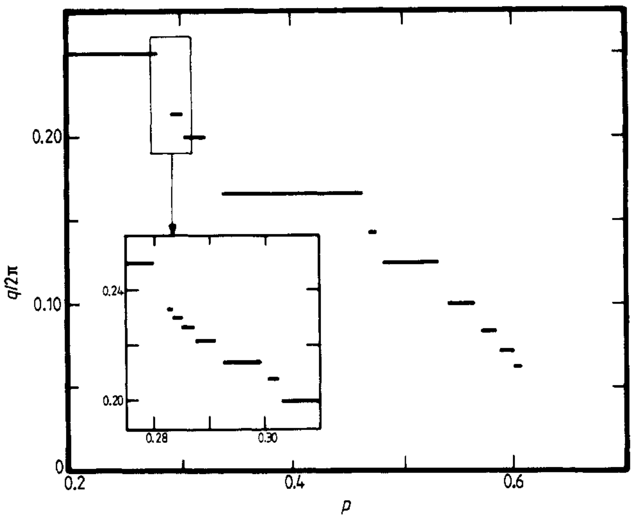

Figure 2. Devil's staircase for $T = 0.2$. Only plateaux with widths $\Delta p \geq 0.006$ are plotted. The insert is a magnification showing plateaux with $\Delta p \geq 0.0006$.

the possible existence of incommensurate phases at lower temperatures, we have numerically determined, at a given temperature, all the commensurate structures associated with plateaux wider than a certain minimum length. Let us consider the interval $a(T)$, shown in figure 1 for $T = 0.5$, and calculate the total length, $S(\varepsilon)$, occupied by all the commensurate phases associated with a plateau wider than ε. The difference $\tilde{S}(\varepsilon) = a(T) - S(\varepsilon)$ corresponds to an interval which is occupied by either incommensurate phases or by commensurate phases associated with plateaux of widths smaller than ε. In the limit $\varepsilon \to 0$, we have $\tilde{S}(\varepsilon)/\varepsilon \sim (1/\varepsilon)^D$, where D is the Hausdorff dimensionality of the set of intervals which are still remaining after the subtraction of the commensurate plateaux (see, for example, the calculations of Jensen *et al* (1983) for the devil's staircase of the circle map and the calculations of Yokoi and de Oliveira (1985) for the chiral Potts model). In figure 3 we show some results of these calculations for different temperatures. The slope of the log-log plots gives the values of D, which are plotted in the inset as a function of temperature. For $\varepsilon < 10^{-4}$, all numerical data are on a straight line, and the devil's staircases are self-similar within the accuracy of our calculations. At low temperatures, as $D < 1$, the staircases are complete and there is no room of finite measure for incommensurate phases. The complementary set to the commensurate intervals may be interpreted as a Cantor set of fractal dimensionality D.

As shown in the inset of figure 3, the fractal dimensionality increases with temperature. This non-universal behaviour of D has also been found by Yokoi and de Oliveira (1985) in the case of the chiral Potts model. For $T > 0.5$, however, the calculations are much harder, since the widths of the plateaux associated with the commensurate phases become much smaller. It is then quite difficult to find a precise value for the temperature above which $D = 1$, the devil's staircases become incomplete and the incommensurate phases occupy an interval of finite measure. To confirm the presence of commensurate and incommensurate structures, we have also performed detailed calculations of the Lyapunov exponents associated with the mapping for different values of the parameters. Unlike the case of the analogue of the ANNNI model

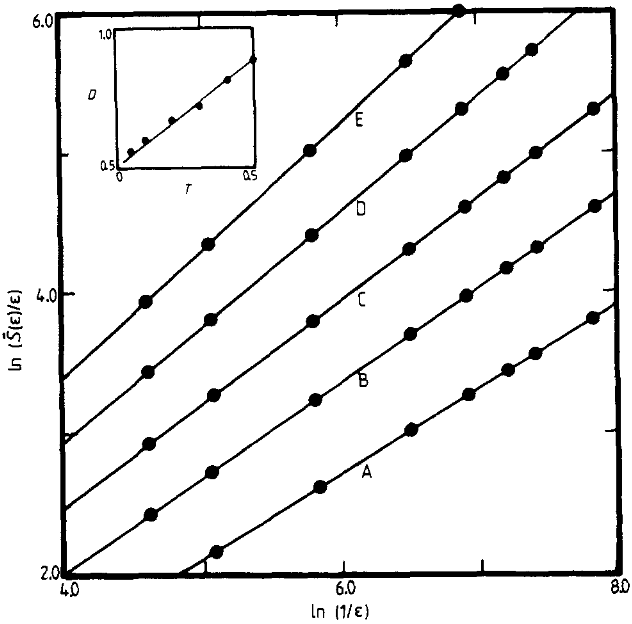

Figure 3. Plot of $\ln(\tilde{S}(\varepsilon)/\varepsilon)$ against $\ln(1/\varepsilon)$ for various temperatures $T=0.1$ (A), 0.2 (B), 0.3 (C), 0.4 (D) and 0.5 (E). For fixed temperature T, the fractal dimensionality of the devil's staircase is given by the slope of the corresponding curve. The insert shows the fractal dimension D as a function of temperature.

on a Cayley tree (Yokoi *et al* 1985), we have not found any evidence of the presence of strange attractors.

TT acknowledges a fellowship from FAPESP, Brazil.

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